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KANT ON MATHEMATICAL DEFINITION*

I

I.1. DEFINITION AND THE THEORY OF SCHEMATISM

What does Kant mean by 'definition'? Strictly speaking,

(*) to define is "to present the complete, original concept of a thing within the limits of its concept" (A727 = B755).

That is to say, the concept, must be presented with all the characteristics (*Merkmale*) which are sufficient for its distinction and which can be seen to belong to it directly, i.e. without any proof. In other words, the *original* concept must be presented with *completeness* and *precision* (A727 = B755 footnote; *Logik* §99). According to Kant only a few concepts are definable in the sense of the quoted statement (*) and these few are mathematical concepts.

However, throughout the *Critique*, definition is linked to the notion of meaning. For instance, in speaking of categories, Kant maintains that they are ways of giving an object "its *meaning* [...] on the basis of some functions of the understanding, i.e. of *defining* it" (A245. Italics mine). This raises the problem of how the sense (*) of definition fits into Kant's theory of meaning, for surely he did not believe that concepts which are not definable in the sense (*) are meaningless. To solve the problem, we shall start by explaining what giving a meaning to a concept represents for Kant, and then we shall compare the sense (*) of definition with such an explanation.

Kant says that a concept has a meaning when "it refers to an object" (B303), i.e. when it is "made real" through a reference – either *a priori* or *a posteriori* – to an intruition. The way such a reference is established is described in the theory of schematism. (Incidentally, a connection between definition and meaning seems to be admitted by Kant even independently of any reference of concepts to objects, as in the case of the pure "logical meaning" (A147 = B186) of unschematized categories. This question will be dealt with later on.)

Maria Luisa Dalla Chiara (ed.), Italian Studies in the Philosophy of Science, 423–452.
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The theory of schematism, although apparently devised to explain how categories refer to intuitions, is an all-inclusive theory of the relations which concepts in general have with intuitions (and vice versa). As is well known, schematism is a procedure of the faculty of imagination (*Einbildungskraft*) meant to provide concepts with schemata that can act as intermediaries between them and "images", i.e. intuitions. Schemata and images are carefully distinguished by Kant. A schema is an *a priori* determination of time-intuition that depends on the "productive" *Einbildungskraft* and relates concepts to images; an image, on the other hand, depends on the "reproductive" *Einbildungskraft* and is the end-product of the whole procedure of schematization.

Not only does this general theory not ignore the differences between the various types of concepts, but it even enhances them by making it easier for us to understand what differentiates definable concepts from all the others. In fact, while every schematizable concept has a meaning, schemata are not all of the same kind and only some of them can guarantee a strict definition of the concepts they refer to.

Thus "the schema of a pure concept of understanding [the transcendental schema of a category] can never be brought into any image whatsoever" (A142 = B181). This means that categorial concepts, although capable of referring to objects, and hence meaningful, have no completely and precisely determinate meaning. Therefore, in the strict sense (∗) of definition, categories can have an 'exposition' but never a 'definition' (A729 = B757; A244 – 5).

As regards empirical concepts, Kant is so certain of their meaningfulness that he maintains that "many empirical concepts are employed without question from anyone. Since experience is always available for the proof of their objective reality, we believe ourselves, even without a deduction, to be justified in appropriating to them a meaning, an ascribed [*eingebildete*] significance" (A84 = B117). But empirical concepts pose a problem opposite to that posed by categories: their Schemata can be reduced to innumerable images, but such schemata bear too close a resemblance to common empirical examples and therefore cannot achieve precision and completeness greater than that obtained by exemplification.[1] So it is easy to understand why empirical concepts fail to satisfy sense (∗)'s requirements for definition and are only capable of a "designation" that makes them explicit but never properly defined (A728 = B756; *R* 2917).

Finally the ~~*Schematismuskapitel*~~ deals with "pure sensible concepts"

which, judging from Kant's examples, are concepts of figures in space, i.e. geometrical concepts. Their schemata are "a product and, as it were, a monogram, of pure *a priori* imagination, through which, and in accordance with which, images themselves first become possible" (A141 = B181). Now, it is the monogramatic nature of these schemata that makes it possible to attribute a complete and precise meaning to pure sensible concepts. In fact, such schemata, unlike those of pure concepts, can be reduced to images; and such images, unlike those of empirical concepts, are completely 'prefigured' by the *typus* of their monogramatic schemata.[2] Therefore pure sensible concepts are the only ones to be strictly definable among meaningful concepts.

This shows that Kant's theory of definition is consistent with his theory of meaning, which in turn is modelled on the division of concepts into (*a priori* or *a posteriori*) *given* concepts and *made* concepts (*Logik* §4). Within this framework, sense (∗) of definition fixes a set of conditions satisfiable only by concepts that are not given "either by the nature of understanding or by experience" (A729 = B757), but are arbitrarily (*a priori*) made, i.e. mathematical concepts. Then one might conclude that it is their specific *origin* that distinguishes concepts definable in sense (∗) from all the others.

This conclusion, however, does not really explain how mathematical concepts differ from other concepts with respect to what they have *in common*, i.e. searching for and finding a meaning. Furthermore it remains to be seen why Kant speaks also of 'nominal definitions' and, in referring to categories, as we have already mentioned, of a 'logical' meaning. Therefore it seems necessary to go deeper into his theory of definition so as to disentangle these questions which are not merely questions of terminology.

I.2. THE ANALYTIC-SYNTHETIC AND THE NOMINAL-REAL DISTINCTIONS OF DEFINITIONS. EXTENSION AND INTENSION. CONNECTION BETWEEN THE TWO DISTINCTIONS OF DEFINITIONS

To start with, Kant is perfectly aware of the difficulties met by any theory of definition that has to make use of the German language, which has a single term (*Erklärung*) to indicate 'definition', as well as 'exposition' and 'explication' (A730 = B758). Nevertheless, following a deeply rooted philosophical tradition, he does give a classification of definitions in his lectures on logic, namely in Jäsche's *Logik* and in other *Vorlesungen* on the same subject.

Specifically, definitions are said to be (1) either analytic or synthetic, and (2) either nominal or real.

Analytic Definitions are those of *a priori* or *a posteriori* given concepts (*Logik* §§100, 101).

Synthetic Definitions are those of *made* concepts (*concepti factitii*) (*Logik* §§100, 102). But, since concepts made by an empirical synthesis of given phenomena are not properly definable (*Logik* §103), synthetic definitions are possible only for concepts made *a priori* by means of a construction, i.e. mathematical concepts.

Obviously the above distinctions between definitions is a faithful replica of the classification of concepts in terms of their *origin* and as such, does not say more than what we already knew from our previous analysis.

The distinction between nominal and real definitions is a more complex one, due to the many senses and functions the terms 'nominal' and 'real' have in Kant's statements on definitions. I have singled out three of them for each of the terms in question:

Nominal Definitions: (I) contain the meaning arbitrarily given to a certain name (*Logik* §106), i.e. give a "conventional meaning" to a name (*R* 3006);

(II) are those whose concept sufficiently indicates all that one *subjectively* knows of a certain thing;[3]

(III) indicate only the logical essence (*logische Wesen*) of the concepts they define (*Logik* §106; *R* 3966).

Real Definitions: (I) like all definitions perform the primary function of 'giving a name' to concepts in order to distinguish them one from the other (*Logik Philippi*, p. 460);

(II) are capable of expressing *all* that *objectively* belongs to the defined concepts;

(III) contain the real essence (*Realwesen*) of the defined concepts and are sufficient for the knowledge of the object (*Object*) according to its internal determinations (*Bestimmungen*) insofar as they exhibit the possibility of the object (*Gegenstand*) on the basis of its internal characteristics (*Merkmale*) alone (*Logik* §106; *R* 3006).

Since nominal and real definitions in sense (I) amount essentially to the same thing, let us start directly with sense (II). Here Kant is clearly considering definitions from the viewpoint of the relation between the subject and the object of knowledge and therefore from the viewpoint of the *Materie* of the definienda. This means that a concept that is *given* to oneself can have a nominal definition, for it is sufficient that its definiens con-

tain what one *subjectively* knows of the concept, i.e. its *form*. But since one cannot make explicit *all* that *objectively* belongs to a concept, unless one has actually *made* it, a real definition is possible only for a concept whose definiens was *originally* thought as belonging to it. In a case like this, however, a nominal definition will coincide with a real one.

It may be easily seen that here Kant distinguishes once again between definitions by appealing to the *origin* of the definienda: *a priori* (*RR* 2914, 2918, 2926, 2995) or *a posteriori* (*RR* 2934, 2936, 2945; *Logik Philippi*, p. 460) given concepts can be only nominally defined, while *a priori* made concepts can have real definitions.

In order to appreciate the distinction between nominal and real definitions in sense (III) above, we must recall some aspects of Kant's theory of concepts. This theory, in accordance with traditional logic, arranges concepts in a hierarchical order in which subordinated concepts are the extension (*Umfang*) of higher concepts and higher concepts are the intension (*Inhalt*) of subordinated concepts.[4] Consequently any higher concept is contained *in* the subordinated concepts (as in the case of 'metal' which is contained in 'gold', 'copper', 'silver' etc.) while it contains them *under* itself (*Logik* §7).

Now the logical essence, which is dealt with by nominal definitions in sense (III), corresponds to the *Inhalt* of a concept inasmuch as the latter contains all the necessary *Merkmale* (i.e. the characteristics that must always be found in the represented thing; *Logik*, Ak. IX, 60–61) of that concept. Therefore a nominal definition characterizes the definiendum in terms of the *essentialia* that are its intension. To do this, no intervention of (either *a priori* or *a posteriori*) intuition is necessary, for all that is in question in a nominal definition is the logical aspect of the concept. Thus, when Kant speaks of a logical meaning independent of intuition of *a priori* given concepts (categories) he has in mind their logical essence, which can be the object of a nominal definition. In this sense we can explain how definition and meaning are linked also from a purely logical point of view, thus clarifying the expression, 'logical meaning', mentioned in I.1.

A real definition in sense (III) aims instead at the *Realwesen* of a concept and has to explain its *possibility* on the basis of the internal *Merkmale* alone.

Now, to know the *Realwesen* of something means to know the predicates on which (as principles of determination) depends all that belongs to the existence of that thing.[5] Therefore a real definition is such as to

enable one to decide, when confronted with any object, whether that object falls under the concept of the definiendum or not. Thus a real definition allows one to use the defined concept as an *Erkenntnisgrund* with respect to all concepts constituting its extension. Obviously empirical concepts cannot be so determined as to make them applicable (as principles of determination) to the *totality* of their extension, since one cannot determine completely and precisely the bounds of this extension. In short, one cannot state a rule that allows one to decide, for example, whether, for any possible x, 'x is a metal' is true or false.

Would it then be right to conclude that only non-empirical concepts have real definitions? The answer is only partially yes. Kant in fact requires of real definitions that they show the *possibility* of the *definienda* with respect not only to their form but also to their *Materie*. This restricts the range of real definitions to *a priori* made concepts, which are the only ones whose possibility *der Materie nach* can be shown simply on the basis of their internal *Merkmale*, without any appeal to actual experience. This means that really defined concepts also have a kind of *existence* that can be established *a priori*. For, in every real definition the definiens contains a clear mark by which the "objective reality" of the definiendum is proved beyond doubt (A242, footnote). Thus a real definition (1) enables us to *recognize* all that belongs to the definiendum as the range of its application, i.e. as its *Umfang*; (2) actually *produces* the concept it defines. Clearly this production is not to be understood as the production of a particular instance of the concept, but rather, by involving only *pure* intuition, as the setting of a rule for producing (or recognizing) any of its instances.

All this provides conclusive evidence for the insufficiency of the origin as a criterion for judging the objective reference of the defined concepts. Luckily, though, this does not create any opposition between the analytic-synthetic and the nominal-real distinctions between definitions. On the contrary such distinctions are closely related, for nominal definitions indicate the logical essence of concepts only on the ground of an analysis of their *Inhalt*.[6] On the other hand, analytic definitions of given concepts are nominal in sense (II) and (III): in sense (II), because the analysis of either the coordinated or the subordinated *Merkmale*[7] of given concepts results in what one *subjectively* knows about them; in sense (III), because no analysis of *a priori* given concepts can lead to their *Realwesen*, while any analysis of *a posteriori* given concepts is aimed only at their logical *Inhalt* (if one analyses the *Merkmale* which are subordinated *a parte post*).

Similarly real definitions are possible only for *a priori* made concepts,

i.e. for synthetically definable concepts. Notice that in this case the two distinctions between definitions actually reduce to nothing. An *a priori* made concept has not only a real but also a (complete and precise) nominal definition: for, in "arbitrary concepts" what is subjective and what is objective coincide (*R* 4016), and both synthesis and analysis of coordinated or subordinated (*a parte ante* as well as *a parte post*) *Merkmale* are complete and precise.[8] Conversely, a synthetic definition is possible only for those concepts whose subordinated and coordinated *Merkmale* can be exhaustively analyzed. In short the peculiar capability of *a priori* made concepts to have real and synthetic definitions enables them to satisfy our sense (*)'s requirements for definitions even when they are analytically and nominally defined. For this reason synthetic and real definitions have a special status and, in certain contexts, seem to be the only ones to 'deserve' the name of definitions.

All this leaves no doubt that Kant's attention to the *Inhalt* and *Umfang* of concepts goes beyond a dutiful respect for traditional logic, since it stems from his interest in *meaning* considered as the objective reference of concepts (as is especially clear from real definitions in sense (III). It is in this perspective that real definitions become 'the ideal', as it were, which any definition tries to approach but only definitions of *a priori* made concepts can reach.[9]

I.3. BACK TO SCHEMATISM. THE SPECIAL STATUS OF MATHEMATICAL DEFINITIONS

We have seen what Kant means by 'nominal definition' and by 'logical meaning' of concepts. Thus we are left with only one of the questions mentioned at the end of Section I.1, namely how concepts definable in sense (*) differ from the other meaningful concepts with respect to the way they establish their own meaning.

To find an answer to this question, we shall briefly reconsider schematism from the viewpoint of real definitions. It is obvious that real definitions, involving both concepts and intuitions, are the result of a schematization. It is not so obvious, however, that they indicate the meaning of the definienda not only through, but also *in*, their schemata: a real definition, as we have seen above, corresponds to the setting of a rule (a schema) to which *all* instances of the definiendum must conform, rather than to the production of any of its instances.

As regards empirical and *a priori* concepts, we have already shown that

their undefinability in sense (*) depends on the example-like nature of the schemata of empirical concepts and on the inability of transcendental schemata to be reduced to particular images. Now we can say with more precision that what makes it impossible to define such concepts is the impossibility of defining them *in* their schemata. Conversely, completeness and precision of real definitions are a consequence of the schematic nature of their objective reference or, as Kant says, of the "schematic construction" (*Entdeckung*, Ak. VIII, 192 n.) made by the subject.

However, the priority of meaning over origin as a criterion for classifying definitions is no reason to underestimate the latter's importance, for both criteria are grounded in the same theory of concepts which provides their vocabulary as well as the evidence for their connection. We simply want to stress that, given this common background, the priority of the meaning-criterion is the true Kantian contribution to the theory of definition.

This is largely a result of Kant's ever present reflection on the nature of mathematics and, in particular, on mathematical definition. If references to mathematical definitions have been kept to a minimum in the preceding pages, this is not only because enough emphasis has been put on Kant's identification of sense (*) with the mathematical sense of definition, but also in order to avoid some necessary distinctions at this stage within the sphere of mathematical definition (see Part III). Moreover all that really needs to be stressed at present is that mathematical definitions can be established as the model on which Kant's theory of synthetic and real definitions has been built up, so much so that the growing importance of the meaning-criterion can be fully appreciated through an analysis of the development of Kantian thought on mathematical definitions from some of his earlier writings to his major work. In Part II we shall briefly do just this.

II

II.1. EARLY CLASSIFICATION OF CONCEPTS AND DEFINITIONS. THE ORIGIN-CRITERION

The best evidence that the meaning-criterion is the *novum* of the *Critique* is the fact that in earlier writings, origin was Kant's sole criterion for classifying concepts and definitions. The so-called *Prize Essay*, published in 1764, begins in fact by stating that mathematics always forms and de-

fines its concepts by means of an "arbitrary connection" of concepts (Ak. II, 276).

This statement has often been interpreted in two contradictory ways: either as a forerunner of Kant's critical doctrine of mathematical definition in A713 = B741 or as an *unicum* Kantian thought in showing a conventionalist-combinatory approach to definitions in mathematics.

Both interpretations miss the point. The *Prize Essay* is a forerunner of the *Critique*, but only insofar as it maintains that mathematical concepts are *made* concepts and that their definitions are, accordingly, synthetic. However, it does not contain anything similar to the critical idea that mathematical concepts must be *constructed*, in the sense that one should exhibit *a priori* the intuition corresponding to them (A713 = B741). The *Prize Essay* lacks a theory about the objective reference of concepts, i.e., it lacks precisely the meaning-criterion.

On the other hand the *Prize Essay* is not so different from the *Critique* as to justify its being considered as a unique concession to conventionalism in Kant's philosophical career. In the first place Kant's anti-conventionalism in mathematics is one of the very few unchanging features of his thought, whatever the stage of his philosophical development.[10] In the second place the term '*willkürlich*', as L.W. Beck rightly points out, does not mean 'random',[11] but rather 'independent from experience'. Finally, Kant speaks of an arbitrary connection of *concepts*, which means that some well-individuated concepts are *given* as the only ones an 'arbitrary' connection may start with. Kant goes so far as to mention these concepts: they are the concept "of quantity in general, of unity, of plurality, of space . . ." that must be assumed for *allgemeine Arithmetik*, *Zahlwissenschaft*, *Geometrie* (Ak. II, 279). Thus the arbitrariness of connection is restricted to a content-sphere, precisely determined by a set of given concepts and by a number of fundamental propositions (*Grundsätze*) expressing their general properties and regulating their use.[12]

However, it has been claimed that the conventionalist-combinatory interpretation is supported also by the *Prize Essay*'s mention of the capability, specific to mathematics, of representing concepts *under*, and not only through, signs (Ak. II, 278) and of operating with them. This is certainly so, but one should not overlook Kant's care in explaining that (1) between geometrical concepts and their representations there is a mutual relation of similarity (*Ähnlichkeit*; Ak. II, 292); (2) arithmetical (*Buchstabenrechnung's*) signs are sufficient for certainty (ibid.); (3) algebraic symbols must undergo an interpretation (Ak. II, 278). Moreover all

these signs are "visible" (*sichtbare*, Ak. II, 279), i.e. sensible, signs carrying with them the immediate evidence of sensibility, especially when they are similar to the concepts they represent.

Now, apart from contradicting the conventionalist interpretation, does all this (the restriction of content and the connection with sensibility) amount to a meaning-criterion in disguise? I do not think so. The question of meaning is surely covertly present in the *Prize Essay*, but the latter does not contain (nor does it need) a proper meaning-criterion. On the one hand, the restrictions on the content of mathematical definitions play the role of formation-rules of mathematical concepts and, as such, constitute a criterion concerning their *origin*. On the other hand, the metaphysical background of the *Essay*, as we shall see, is sufficient to guarantee the meaningfulness of arbitrarily defined concepts. Therefore there is no need to specify rules for assigning a meaning to concepts in general and to mathematical concepts in particular. Thus, there being no *rules*, there is no *criterion* to speak of.

To better understand this situation it may be useful to look at another essay of the same period where, in a much wider context, Kant investigates the modal status of mathematical concepts; the *Beweisgrund*. There he claims that mathematical concepts belong to the sphere of possibility since they are neither 'necessary' in themselves, nor actually 'existing' (Ak. II, 75). It is easy to see that this agrees perfectly with the *Prize Essay*'s thesis that mathematical concepts are neither 'given' nor 'found', but are made by their definitions.

Kant's next step is to analyze some of these concepts, coming to the following conclusion: in all possible concepts one can distinguish *das Formale* (or *das Logische*) and *das Materiale* (or the *data*) of possibility (Ak. II, 77–78). On this ground he argues for the dependence of possibility on existence, maintaining that, should *all* that exists be taken away, nothing would be left, not even the *data* for possible concepts like the mathematical ones (Ak. II, 78).

Obviously Kant does not mean that, in order to make (or to define) a possible concept such as that of a right triangle, its *Materialia*, namely 'right angle' and 'triangle', should actually exist. He simply says that something must be supposed to exist and that, in the case of geometry, this is space (Ak. II, 80–81). Thus, while from the viewpoint of the *Prize Essay*, it is sufficient to consider space as a primitive, from the viewpoint of the *Beweisgrund* it is necessary to attribute to space a kind of existence.

Therefore the connection between mathematical concepts and existence

can be traced back to the primitive terms of mathematics and to universal propositions about them. Thus we know that and where the question of meaning is at work at this stage of Kant's philosophy, but we cannot say that he has a full-fledged theory about it.

II.2. FROM THE ORIGIN-CRITERION TO THE MEANING-CRITERION

The priority of existence over possibility has, from a gnoseological point of view, quite a few consequences, the most interesting being that the way one comes to know what exists, i.e. experience, must have a privileged status among all *modi cognoscendi*.

Once this autonomous result of Kant's research on the theme of knowledge came into contact with the influence of Hume in the late sixties, a period of radical empiricism was almost bound to ensue.

I shall not try to give a reconstruction – detailed or otherwise – of this period. Nevertheless I believe that the next few remarks will at least hint at the reasons why this period marked the beginning of the emergence of meaning as a criterion.

The sudden elimination of metaphysics from Kant's philosophy, caused by his empiricist 'crisis', made it impossible to take for granted any longer the connection between mathematics and the existing world. And this applied to primitive and derivative terms alike. Kant then, in order to safeguard the objective reference, the meaningfulness, of mathematics did not hesitate to link it directly with existence by means of experience: mathematics renounced its apriority (*RR* 3750, 3738, 3923).[13]

What is most relevant here, apart from the extravagance of the solution with respect to the critical doctrines, is that the objective reference of mathematical concepts had gained such an importance as to make Kant (temporarily) overlook his own assertions about the non-empirical character of mathematics.

As is well known these – for Kant – extremist views were to be abandoned mostly because the sure possession of a meaning *via* induction was counterbalanced by the loss of universality of mathematical concepts as well as of mathematical-physical laws. Many intellectual "capsizings" (*Umkippungen*), recorded in the *Nachlass*, were still to be brought about by tentative solutions of this problem before the revolution that led to the *Critique*.[14] The importance of meaning as a test for all concepts, however, was to remain unquestionably established.

Surely the origin-criterion regained its value, but only after it had been

deprived (at least with respect to mathematical concepts) of any 'combinatory' connotation it might have had in earlier times. This was possible because Kant had finally stated the *rules*, i.e. the criterion, for the meaning of concepts in the – unfortunately too short – *Schematismuskapitel*.

III

III.1. MATHEMATICAL DEFINITION FROM A SYSTEMATIC VIEWPOINT: SOME OPEN QUESTIONS

In Part I and Part II we have dealt with Kant's theory of mathematical definition in general. Now we shall discuss a suspicion of one-sidedness that can be raised about the theory itself: it is a fact that in most critical and pre-critical writings, Kant refers to definitions of geometrical non-primitive terms and acknowledges a special status for monogramatic schemata of geometrical concepts.

This suspicion of a geometrical bias is all the more perplexing if confronted with Kant's clear awareness of the systematic character of mathematics. For he conceives mathematics as a system with respect to (1) its subject-matter, and (2) its internal structure. From both these viewpoints no privileged position is assigned to geometry, nor *a fortiori* to definitions of its derivative terms.

We can substantiate Kant's systematic view of mathematics as regards its subject-matter with evidence provided by the notes of his lectures taken by Herder in the years 1762–1763. The two series of notes that constitute the mathematical part of Herder's *Nachschriften* agree in indicating quantity in general as the subject-matter of mathematics, and in distributing the matter as indicated in Table I.[15]

The slight differences between the two Series are easily overcome since the subdivision of the Mathesis universalis into Series I corresponds to that of Series II if one drops common calculus and all parts of analysis except algebra. Likewise the subdivision of the Mathesis specialis into Series I corresponds, in its 'pure' part, to the subdivision relevant to space in Series II, geometry being precisely the special part of mathematics dealing with pure space. All this shows that the different parts of mathematics are organically related to each other so that all Kantian assertions about mathematics, including those about definitions, should be meant to refer, unless otherwise stated, to the whole of mathematics.

As regards the systematic aspect of mathematics from the viewpoint of

TABLE I

	Series I	Series II
Mathesis Universalis	(1) Common Calculus (2) Higher Calculus (A) Buchstabenrechnung [algebra of integers] (B) Analysis: Algebra of equations Calculus of infinity	(A) Arithmetic ("durch Zahlen") ['through numbers'] (B) Algebra ("durch andre Zeichen") ['through other signs']
Mathesis Specialis	(1) "Pura" (A) pure geometry (B) trigonometry (2) "Impura sive applicata" ['impure or applied']	(1) of space: geometry trigonometry (2) of time (3) of force

its internal structure, Kant unquestionably assigned a *deductive* character to all mathematical disciplines, independently of their capability to have 'axioms'. Therefore he could not be unaware of some of the problems concerning deductive systems, such as: Are their primitive terms introduced by definition or otherwise? What is the relation between the primitive terms and the principles of such systems?

To find an answer to these questions, and in general to dispel the suspicions of one-sidedness about Kant's theory of mathematical definitions, we shall deal separately with each part of pure mathematics according to the above subject-matter division.

III.2. MATHESIS UNIVERSALIS

Mathesis universalis (*allgemeine Grössenlehre*) is the science of quantity (*Grösse, quantitas*) and of the operations that can be performed on quantity. Its primary aim is to provide a set of universal fundamental propositions valid for all quantitative sciences. It is noteworthy that Kant does not speak here of axioms or 'principles' (*Prinzipien; Logik* §36 Anmerkung 2), but only of *analytische Grundsätze* that, according to the examples he gives, correspond to Euclid's *κοιναὶ ἔννοιαι* [common notions].[16] Their analyticity makes them flexible with respect to interpretation: an analytic *Grundsatz*, such as 'the whole is greater than the part', is valid for Kant (i.e. originates a true proposition) whatever its *quantitative* interpretation because it does not require any preliminary 'translation', e.g. into arithmetical terms. Obviously, once interpreted, all such propositions cease to be

analytic and fundamental and become synthetic *a priori* propositions of the specific branch of mathematics in whose terms they have been interpreted.

However, *allgemeine Grössenlehre* has another very important function: it is also an *allgemeine Arithmetik* (Ak. II, 282) insofar as, according to *H. Nach*, it contains algebra as one of its parts. In this sense it constitutes the instrument of any arithmetical generalization (see section III.4) and, by substituting *Zahlen* for its *Zeichen*,[17] becomes a synthetic *a priori* science. This special relation with arithmetic can be best explained by reference to definitions.

The first thing to be noticed in this respect is that the subject matter and the only primitive term of algebra is (as in the case of *allgemeine Grössenlehre*) indeterminate quantity (Ak. II, 287). This is a mathematically undefinable concept since, according to the *Prize Essay*, it is an unanalyzable concept to be assumed in mathematics without definition (Ak. II, 279); according to the *Critique* it is a category.

However, the concept of quantity is schematizable and hence meaningful, but, given its categorial nature, its schematization is a process which is far from being straightforward. In fact quantity "seeks its support and sense in number; and this in turn in the fingers, in the beads of the abacus or in strokes and points which are placed before the eyes" (A240 = B299). Thus two steps are involved here: from categorial quantity to concepts of number (i.e. to concepts of quantity determinate with respect to unity); from concepts of number to sensible intuition. Both steps entail the number-schema, "a representation which comprises the successive addition of homogeneous units" (A142 = B182). But while the whole process, necessary to indicate the objective reference of quantity, does not define it, the second step is sufficient to produce the determinateness of strict definition. Of course, since this determinateness depends on the fact that the number-schema does not apply directly (as a transcendental schema) to the category, but rather to concepts of number, the latter, and not *unbestimmte* quantity, are capable of strict definition (see Section III.3).

As for algebraic operations, Kant does not define them but simply mentions some "easy and certain rules" (Ak. II, 278) that regulate their use. He does not state these rules clearly but refers to them as if they were able to define algebraic operation-signs implicitly. Obviously such implicit definitions do not have sufficient precision and completeness and therefore are not strict definitions. Furthermore the rules that constitute these implicit definitions are analytic and can be assigned an objective

reference only through an interpretation. Indeed they require an arithmetical interpretation: they have to be given numerical values if they are to be found true and synthetic *a priori*.

This is why, insofar as it is an algebra, *allgemeine Grössenlehre* has a special relation with arithmetic, so much so that also a rule like "*Gleiches zu Gleichem addiert gibt gleiche Summe*", which is proved by purely algebraic means (*H. Nach*, p. 23) and is independent of the definition of addition of specific numbers (cf. section III.4), cannot be synthetic unless it is assigned an arithmetical meaning by transforming references to indeterminate quantity into references to determinate quantities.[18]

Now, since insufficient determinateness is responsible for the undefinability in sense (*) of quantity, we must ask whether arithmetic proper (*Zahlwissenschaft*), which is the science of quantity determinate with respect to unity, admits synthetic and real definitions and, if so, what are and how are the primitive terms introduced which enter into the definitions of numbers.

III.3. ARITHMETICAL DEFINITIONS. NUMBERS AND PRIMITIVE TERMS

The schematization of quantity considered above shows that number is a concept that belongs to the category of totality (B111) and synthesizes (relates) unity and plurality. Therefore numbers, being the result of a relation, i.e. of an intellectual act, are intellectual concepts: as Kant says, number is a concept "se quidem intellectualis" (*Inaugural Dissertation*; Ak. II, 397). But have we not seen that concepts definable in sense (*) have monogramatic schemata and therefore are pure sensible concepts?

Let us recall Kant's requirements for definability as summarized in A729 = B757: "there remain, therefore, no concepts which allow of definition except only those which contain an [1] arbitrary [2] synthesis [3] that admits of *a priori* construction". Now concepts of number satisfy:

Requirement [1]: "no other purely arbitrary concepts of pure reason can rise in us but those [obtained] through repetition, consequently concepts of number and quantity" (*R* 3973);[19]

Requirement [2]: "our counting, as is easily seen in the case of larger numbers, is a synthesis according to concepts, because it takes place according to a common ground of unity as, for instance, the decade" (A78 = B104);

Requirement [3]: in any rational knowledge "only relations are to be

considered, and these are either given [...] or made up [*gedichtet*]. But we can make up no other relation whose possibility we can be sure of, but those made up according to quantity through repetition in *Zahlwissenschaft*" (*R* 3940). Moreover "our reason contains nothing but *relationes*. Now, if these are not given through the relations according to space and time in experience, or through the repetition and the synthesis of many into one in pure mathematics, then these are not *relationes* that refer to objects but only *relationes* of our concept according to our reason" (*R* 3969).[20]

All this amounts to saying that the concepts of number are *a priori* made (*gedichtet*, *willkürlich*) and therefore satisfy the origin-criterion for synthetic definitions. At the same time the emphasis on 'repetition' and on 'synthesis of many into one', as the only relations (apart from those given in experience) which refer to objects, shows that the *Zahlwissenschaft's* concepts also satisfy the meaning-criterion for real definitions. In short, concepts of number may be constructed because their *Inhalt* as well as their *Umfang* can be completely and precisely determined by our schema of quantity (the number-schema), just as happens for pure sensible concepts, thanks to their monogramatic schemata.

This does not mean that there are no differences between concepts of number and pure sensible concepts. When it comes to definition, definitions of numbers, unlike geometrical definitions, must always be technically *genetic*, i.e. they must show the rule that generates them also in their form. This depends on the fact that the number-schema, unlike monogramatic schemata, is not only the same for all number-concepts, but also provides them with a meaning in one way only (A165 = B205 – 6), namely by considering them under the form of *one-dimensional* time.

It is worth noticing that Kant's early works already contain, in addition to a definition of number that echoes the traditional Euclidean one,[21] a definition founded on the repetition of unity. But, while in the pre-critical period the notion of repetition seems to depend entirely on the notion of measure,[22] in the critical period the notion of repetition becomes the element that – within schematism – determines the reality of the definitions of numbers using, rather than depending on, the number-measure relation. This supports our thesis that the critical *novum* consists in the emphasis put on the reality of mathematical definitions which partly overshadows the attention paid to the origin of the definienda.

As for the question of primitive terms, it arises as soon as one realizes that, although the definitions of numbers are definitions of *a priori* made concepts, nowhere does Kant say that such concepts are made *ex nihilo*.

On the contrary, concepts of number originate from a synthesis of unity and plurality and so their definitions presuppose unity (not to be confused with the number one) and plurality.

Again, if we look at Kant's pre-critical theories we find that the *Prize Essay* actually indicates unity and plurality (*Einheit* and *Menge*) as the unanalyzable (*unanflösliche*) concepts of arithmetic (Ak. II, 279–80), while *H. Nach* (p. 21), referring to unity, says that in mathematics it must be considered as a "bekandter Begrif".

In the *Critique*, too, these concepts are simply 'assumed', but now for the reason that they are categorial, i.e. *a priori* given, concepts. As such they either have a nominal (and for mathematics useless) definition, or are founded in *a priori* intuition through schematism. In the latter case, since their schema is transcendental, they cannot have real and synthetic definitions, but are nevertheless quantitatively meaningful concepts and therefore can be used in mathematics.

III.4. ARITHMETICAL OPERATIONS. THE QUESTION OF AXIOMS

Zahlwissenschaft is not only the science of numbers, but also the science of the operations that can be performed on numbers. Therefore we must also investigate whether Kant believes them to be strictly definable. Certainly he does not identify such operations with those of the *allgemeine Grössenlehre* since they have no implicit definitions. In fact in *H. Nach* (p. 22) "addiren" is explicitly qualified as the operation that "finds one number (the sum), which is equal to different numbers (of the same kind but *not* of the same magnitude) taken together".[23] Thus arithmetical operations do refer to something precise that, in the case of addition, is the taking together, the synthesis, of different numbers into one. By the way, a synthesis is involved not only in addition but also in subtraction (Ak. XI, 555; *Zweig*, 129) and presumably in all four fundamental operations of arithmetic.

But does this objective reference of arithmetical operations make them strictly definable? Let us consider for instance the case of addition. We may notice that (1) it is an operation schematized by the number-schema since it refers to the counting of units; (2) like the concepts of number, it can determine its *Umfang a priori*, completely and precisely through construction. Therefore the definition of addition as the taking together of units does satisfy the requirements for strict definability. This can be gathered also from the fact that this operation is subject to the restrictions

imposed on numbers by their origin and meaning. For any addition of numbers can take place *in one way only* and no addition of an actually infinite number of addenda (as well as no addition of addenda of infinite magnitude) is possible because it would have no objective reference.

We may generalize this conclusion by saying that the four arithmetical operations are strictly definable. But, for this very reason, they also have a narrow range of application and no universality apart from the universality of use. We must remember in fact that an arithmetical operation for Kant is only an operation performed on *specific* numbers. As he so forcefully maintains, $7 + 5 = 12$ is a synthetic *a priori* proposition, but is also "singular" and a "number formula" (A164 = B204).

This brings us directly to the question why, for Kant, "arithmetic has no axioms" (Ak. XI, 555; *Zweig*, p. 129; A164 = B204). In order to understand the true meaning of this statement we have to examine it in the light of both the relation between *allgemeine Arithmetik* (algebra) and arithmetic, and the influence of intuition on the foundations of arithmetic.

Above all the algebra-arithmetic relation is, as we have mentioned before, a relation of interpretation of the first in terms of the second. It is by interpretation that algebra acquires an arithmetical meaning. In particular arithmetical operations provide a meaning for algebraic operation-signs. But the relation between algebra and arithmetic is a *mutual* relation: on the one hand arithmetical operations give a meaning to the algebraic ones, on the other hand the rules that constitute the implicit definitions of algebraic operations can also be applied to the arithmetical operations. Indeed arithmetic, insofar as it belongs to the general system of mathematics, presupposes an *allgemeine Arithmetik*. In *H. Nach* (p. 27) Kant gives a proof of $8 + 4 = 12$ which makes essential use of the algebraic rule "*Gleiches zu Gleichem addiert gibt gleiche Summe*". Algebra therefore is important for the enlargment of arithmetical knowledge: as Kant says, "algebra is an ampliative science" (Ak. XI, 554; *Zweig*, p. 129).

That the above-mentioned proof of $8 + 4 = 12$ is to be found in a pre-critical text does not detract from the role played by algebra in arithmetic, also in the later critical theory. This is something to be stressed because some of the *Critique's* remarks on $7 + 5 = 12$ have misled many of Kant's readers into believing not only that all such operations require is some counting on one's fingers, but also that either Kant did not have a theory about operations on larger numbers or he considered small numbers (and operations on them) to be very different from larger numbers (and operations on them).

Now Kant has never expressed himself so ambiguously as to justify such misinterpretations. For him there are no differences between small and larger numbers, so much so that, in order to explain the true nature of the number-schema, he refers to the case of larger numbers (A140 = B179). Furthermore, in contrast to his critic, Eberhard, who maintained that the concept of a chiliagon is an idea of reason because it could hardly be immediately represented in intuition, Kant argued that the *number* of sides of a geometrical figure is irrelevant as far as its mathematical nature is concerned "for [...] the construction of the object can be completely prescribed" (Ak. XI, 47; *Zweig*, p. 149).

But if Kant does not distinguish between numbers according to their magnitude, neither does he distinguish between operations to be performed on numbers, whatever their magnitude. This explains how algebra can be the common background for all arithmetical operations. Let us consider for instance an equality like $1000 + 2000 + 1 = 1000 + 2001$. This would appear to be established as a correct equality only by calculating each of its members and then appealing to the following analytic *Grundsatz* of *allgemeine Grössenlehre*: things equal to a third thing are equal to each other. However, the algebraic rule "*Gleiches zu Gleichem addiert gibt gleiche Summe*" allows us to establish the above equality, independently of the computation of its members, by considering it as one of its instances. The possibility of this indirect proof, naturally enough, shows all its importance where larger numbers are involved because it eliminates the difficulty of practical computation.[24]

This does not mean that *Zahlwissenschaft* does have axioms (synthetic *a priori* universal propositions) after all. For as soon as an arithmetical meaning is given to propositions such as "*Gleiches zu Gleichem addiert gibt gleiche Summe*", they cease to be both analytic and universal just like analytic *Grundsätze*. Nevertheless their very existence provides sufficient evidence for a connection between algebra and arithmetic. Such a connection prevents one from attributing to Kant a conception of arithmetic as consisting only of infinitely many, mutually independent, singular number-formulae.[25] On the contrary, to paraphrase a famous Kantian dictum: algebra without arithmetic is empty, arithmetic without algebra is blind.

Finally, the fact that arithmetic has no axioms makes it clear that Kant never considered it to be the science of time in the same sense in which geometry is the science of space: time intuition has no direct influence on the properties of numbers (Ak. XI, 556; *Zweig*, p. 130) and "the objects

of arithmetic and algebra, according to their possibility, are not subject to temporal conditions" (*R* 13). This is perfectly consistent with the Transcendental Aesthetic where axioms of space correspond to geometrical axioms, but axioms of time[26] have nothing to do with arithmetic. Thus the lack of a special object of arithmetic and of axioms about such an object entails, from a structural point of view, what we have seen to be typical of arithmetical definitions from a gnoseological point of view, i.e. the necessity for them to be genetic with respect to their form as well as their content. In geometry on the other hand, as we shall presently see, even non-genetic definitions are acceptable because axioms are sufficient to guarantee the reality of geometrical concepts.

III.5. DEFINITIONS IN GEOMETRY

Geometry is a more specific science than arithmetic because it is the part of pure mathematics that has a special object, namely space and spatial relations (Ak. II, 397, 403; A163 = B204). Now space cannot be mathematically defined: in the *Prize Essay* Kant calls it an *unanflöslich Begriff* to be assumed in mathematics simply according to its common representation (Ak. II, 278); in the *Critique* the concept of *geometrical* space is said to require no philosophical legitimization (A87 = B120). Furthermore, consistent with Kant's general thesis, that *a priori* given concepts can have an 'exposition' rather than a definition, the Transcendental Aesthetic gives an exposition but does not define space. Naturally there are no doubts about space's objective validity and meaningfulness (*objective Gültigkeit* and *Sinn und Bedeutung*) because one can establish its necessary application to the objects of experience (A156 = B195). Therefore, being a mathematically undefinable but meaningful concept, space is the true geometrical *primum*.[27]

However, traditionally, geometrical primitives are identified with notions like 'point', 'line' etc. How does Kant deal with them? The answer to this question emphasizes the difference between quantity and space, and consequently between (universal or particular) arithmetic and geometry. In fact space, inasmuch as it is the object of geometry, "contains more" than the simple form of intuition because it contains the *formal* intuition (*formale Anschauung*) which provides the unity of representation (B161, Note). In other words the object of geometry is not so much the form of our sensibility as the organization of spatial patterns according to such form.

This implies that, while one may properly speak of an indeterminate quantity, one could not speak of an *empty* space. For concepts (and categories) depend on the subject's activity in order to acquire their form, while no intuitive (*a priori* or *a posteriori*) representation is originally given) to the subject independently of its form.[28] Thus, on the one hand, indeterminate space cannot be represented at all, on the other hand, space is necessarily presupposed by all its determinations. In Kant's own words this means that "space should be properly called not compositum but totum, since its parts are possible only in the whole, not the whole through the parts. [. . .] If I remove all compositeness from it nothing remains, not even the point. For a point is possible only as the limit of a space" (A438 = B466–68).[29]

Thus Kant avoids the usual negative definitions of traditional geometrical primitive terms (a negative definition "is not a real definition", *Logik*, §106 *n*, *R* 2916). At the same time he obeys the Aristotelian imperative that one should prove the existence of terms introduced in geometry. For Kant proves the existence of non-primitive terms by means of real synthetic, i.e. constructive, definitions and assumes primitive terms together with their existence. He does so (1) by arbitrarily choosing on which simple parts to build up geometry, since simple spatial parts are not given *per se*; (2) by indicating ostensibly (and so showing the existence of) the chosen parts.[30]

That geometrical derivative terms for Kant have synthetic and real definitions is a well-established fact that needs no further explanation. Rather, it is interesting to consider the question of axioms with respect to definitions.

To start with we may notice that for Kant geometrical axioms "express the conditions of sensible *a priori* intuition" (A163 = B204) and "can be represented in intuition" (*Logik* §35). Furthermore axioms are immediately certain, synthetic *a priori* universal principles (A732 = B760; A299 = B356). Now even a superficial examination of Kant's examples of axioms makes it clear that they correspond to *some* of Euclid's *postulates*. For instance the Kantian axiom "two straight lines do not enclose a space" (A163 = B204) has been proved since Proclus' times to be reducible to Euclid's first postulate.[31] Clearly then Kant considers as axioms (i.e. as *Principia* and not merely as *Grundsätze*) those Euclidean postulates that express general properties of geometrical space and retain the classical function of extra-logical restrictions on the arbitrariness of geometrical definitions. The fact that some Euclidean postulates are not among

Kant's examples of axioms can be explained on the grounds that their originary function as existential assumptions on non-primitive terms is made redundant in the *Critique* by the schematization involved in the definition of concepts. This is evident in the case of Euclid's third postulate.[32]

Now, granted that the impossibility of giving a real definition of a figure enclosed between two straight lines (A220 = B268) can be established simply by acknowledging that it is impossible to construct it (and therefore not necessarily by appealing to the above axiom), the following question could be raised: Could geometrical axioms be dispensed with? The answer is plainly No.

First of all, Kant is perfectly aware that science is not so much concerned with the way its objects are obtained as with the properties of these objects (Ak. XI, 40–48; *Zweig*, pp. 145–6). But the properties of geometrical objects are established by proving theorems about them, and in turn proofs appeal to the general properties of space expressed by the axioms. Secondly, the above question seems analogous to that about the dispensability – from an allegedly 'Kantian' point of view – of propositions like "*Gleiches zu Gleichem addiert gibt gleiche Summe*", granted that numerical propositions, e.g. 1000 + 2000 + 1 = 1000 + 2001, can and should be calculated independently of it. But, similar to what has already been pointed out for arithmetic, universal propositions (and *a fortiori* geometrical axioms) guarantee the constructibility and provability of properties of mathematical concepts that are highly complex and difficult to construct practically. This means that whoever poses the above question is concerned more with the explicit formulation than with the assumption of a set of axioms, for it is evident that he tacitly assumes them in all the 'easy' cases of construction.

What is more, the question reflects a viewpoint which is so little 'Kantian' as to overlook the fact that Kant, although convinced of the Euclidean character of man's intuition (which, because of its uniqueness, would justify the lack of an explicit formulation of the axioms), could certainly accomodate a different geometry within his philosophical framework. Such a geometry (or geometries) would have the nature of a 'game' (A239 = B298; A175 = B196) and would be incapable of application to reality, but still would not be absurd. This cliam is indirectly supported by the fact that arithmetic, which has no axioms, does not allow anything of that kind: there is just *one* arithmetic.[33]

This structural difference between geometry and arithmetic is respon-

sible for the fact that arithmetical propositions are universal only with respect to their use. In fact axioms in general not only express extralogical restrictions on the *Willkürlichkeit* of definitions, but also guarantee the universality of the definienda. Therefore, having no axioms, arithmetic, on the one hand, has to show in each case the possibility of its objects through a technically genetic definition; on the other hand, it cannot state *a priori* the conditions of their intrinsic universality.

Geometry is a different case. As mentioned before, for Kant there is no need to define geometrical concepts in a strictly genetic form: "If a circle is defined as a curve all of whose points are equidistant from a midpoint, is not this concept given in intuition? And this even though the practical proposition that follows, viz., *to describe a circle* (as a straight line is rotated uniformly about a point), is not at all considered".[34] Obviously in the case of complex geometrical concepts a strictly genetic form of definition is used, but this means only that in defining such concepts one follows more closely their schemata. Indeed genetic definitions of complex figures are a matter of practical utility rather than a necessity. What is more, all geometrical definitions, unlike arithmetical definitions, provide their definienda with an intrinsic universality; for instance, once one triangle is given by means of a definition, so are all triangles.[35]

In spite of these differences between geometry and arithmetic, geometry is a science only insofar as it is a part of mathematics, i.e. insofar as spatial figures undergo a quantitative determination. Therefore not only the analytic *Grundsätze* of *allgemeine Grössenlehre*, but also the rules and operations of particular arithmetic, apply to geometry. This is what Kant means when he says that "*Zahlwissenschaft* [. . .] *ist* [. . .] *das Instrument der ganzen Mathematik*" (*H. Nach*, p. 17). Geometry's peculiarity consists in its being a quantitative science whose objects are considered "together with their quality" (A720 = B748). The schemata of geometrical figures are in fact monograms, i.e. 'figurative' schemata.

According to Kant, though, the relation between geometry and arithmetic is neither a mere matter of mathematical architectonic, i.e. of an 'external' connection represented by their sharing the language of numbers, nor does it go in one direction only (i.e. from arithmetic to geometry). In a letter to A.W. Rehberg (Ak. XI, 207–10; *Zweig*, pp. 166–169) Kant emphasizes once again the "necessary synthesis [*Zusammenhang*] of inner sense with outer sense" whenever *quantification* is involved: "time [. . .] has to be imagined as a line, if it is to be quantified, just as [. . .]

a line can be quantified only by being constructed in time". But what is really noteworthy in this letter is that Kant assumes a connection between arithmetic and geometry stricter than the general *Zusammenhang* of spatial and temporal intuitions. For a quantity like $\sqrt{2}$, although not representable as a "complete numerical concept (the relationship of $\sqrt{2}$ to unity)", can have a geometrical representation in the diagonal of a square of side 1.

Thus the 'image' of $\sqrt{2}$ is not simply an (*a priori* or *a posteriori*) *spatial* image, as is the case for 'complete numerical concepts', but is a certain *geometrical* construction. Hence, on the one hand, we may associate the diagonal of the square to $\sqrt{2}$ only because the diagonal is considered *sub specie quantitatis* as a mathematical object (see *R* 13); on the other hand it is the association of the diagonal to $\sqrt{2}$ that makes the latter strictly definable. Without this association $\sqrt{2}$ would still be mathematically meaningful, but not strictly definable since its objective reference would consist in a non-terminating (hence never *completed*) calculation.

IV

CONCLUDING REMARKS

Three general remarks can be added as a conclusion.

(1) Kant's theory of definition agrees with his general view that truth is neither the result of an immediate mind-world *adaequatio* nor the product of a convention. First, this explains why so much care is taken by Kant in stating the criteria for the meaningfulness of concepts: all significant concepts must have an object reference, but this reference has to be mediated by schemata. Now, while all concepts that do not have such a reference are not significant (which does not mean that they are absurd), only mathematical concepts whose objective reference is exhibited *a priori in* their schemata have a strictly defined meaning. Secondly, this also explains Kant's refusal to reduce mathematics to logic: all mathematical concepts must show their objective reference, albeit without any intervention of experience.

The construction of mathematical concepts satisfies these requirements. In this sense intuition in mathematics is not a synonym for 'visualization': the completeness and precision of meaning, which are obtained by construction, are attained by all concepts whose *Umfang* is given in its *totality* by their schemata. This means that constructibility is a property not only of concepts that are visualizable by the 'mind's eye', but also of those

mathematical concepts whose complexity and abstraction could hardly allow any visualization.

(2) The second remark concerns the attention that has been paid in this paper to Kant's theory of concepts as it is expounded in his logical *Vorlesungen*. I believe that this theory can serve not only as a terminological aid, but also as a powerful conceptual tool for deeper insight into Kant's thought in general.

For example, by bringing forward this paper's approach to the theory of mathematical definitions through Kant's theory of concepts, one could reconsider the old theme of synthetic *a priori* judgments from a new perspective. For, granted that every definition is a judgment, every strict mathematical definition is a synthetic *a priori* judgment. Twenty years ago L.W. Beck already moved along these lines, but it seems to me that more attention should be paid to the *Inhalt* and *Umfang* as being responsible for the analyticity and syntheticity of concepts and judgments. In this way an investigation into the theory of concepts would not be a substitute for but a complement to the study of schematism since, in considering the extension and intension of concepts, one not only goes beyond their hierarchical ordering but also looks at the theory (schematism) concerning the way their intension and extension could be known. And this is of great importance for a philosophy that hinges essentially on an epistemology.

(3) The third remark is more pertinent to mathematics. We have seen how the theory of mathematical definitions applies to the different parts and levels of mathematics. This has brought to light how important Kant's assessment of mathematics as a system is for his conception of mathematical knowledge. In fact it constitutes both the starting-point for his reflections on the entire subject and the testing-ground for the results of such reflections. Furthermore all this should put the elementarity of Kant's mathematical examples into the right perspective, since this has given way to speculations about the limits of his actual knowledge of mathematics. As we have seen, the consideration of mathematics as a system emphasizes the quantitative, i.e. conceptual, aspect of all its parts. Therefore, on the one hand, the elementarity of the examples is not the consequence of an alleged necessity to visualize mathematical concepts; on the other hand, as we have seen, non-elementary parts of mathematics (e.g. what at the time was considered to be algebra) and classes of numbers other than natural numbers can be accomodated within Kant's general conception of mathematics.

NOTES

List of abbreviations of references used in this essay.

Ak. (followed by the number of the volume in Roman numerals and by the indication of the page) stands for *Kants gesammelte Schriften*, under the editorship of the Königliche Preussische Akademie der Wissenschaften, Berlin; from vol. 23, under the editorship of the Deutsche Akademie der Wissenschaften, Berlin; from vol. 24, under the editorship of the Akademie der Wissenschaften, Göttingen, (G. Reimer), W. de Gruyter, Berlin, 1902–.

A, B, as usual, indicate the two major editions of the *Kritik der reinen Vernunft*. Quotations are from N. Kemp Smith's English translation (*Critique of Pure Reason*. London, Macmillan, 1968) with minor changes.

Beweisgrund: Der einzig mögliche Beweisgrund zu einer Demonstration des Daseins Gottes, (1963), Ak. II, 63–164.

Entdeckung: Über eine Entdeckung nach der alle neue Kritik der reinen Vernunft durch eine ältere entbehrlich gemacht werden soll, (1790), Ak. VIII, pp. 185–251.

Gedanken: Gedanken von der wahren Schätzung der lebendigen Kräfte und Beurtheilung der Beweise, deren sich Herr von Leibniz und andere Mechaniker in dieser Streitsache bedient haben, nebst einigen vorhergehenden Betrachtungen, welche die Kraft der Körper überhaupt betreffen, (1747), Ak. I, 1–182.

H. Nach: I. Kant, *Aus den Vorlesungen der Jahre 1762 bis 1764. Auf Grund der Nachschriften J.G. Herders*, edited by H.D. Irmscher, *Kant-Studien*, Ergänzungshefte **88**, Cologne, Kölner Universitäts-Verlag, 1964.

Inaugural Dissertation: De mundi sensibilis atque intelligibilis forma et principiis, (1770), Ak. II, 385–420.

Logik: Logik. Ein Handbuch zu Vorlesungen, (1800), edited by G.B. Jäsche, Ak. IX, 1–150.

Logik Philippi: Kants Vorlesungen: Vorlesungen über Logik, Ak. XXIV, pp. 303–496.

Naturgeschichte: Allgemeine Naturgeschichte und Theorie des Himmels oder Versuch von der Verfassung und dem mechanischen Ursprunge des ganzen Weltgebäudes, nach Newtonischen Grundsätzen abgehandelt, (1755), Ak. I, 215–368.

Prize Essay: Untersuchung über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral, (1764), Ak. II, 273–302.

Prol.: Prolegomena zu einer jeden künftigen Metaphysik, die als Wissenschaft wird auftreten können, (1783), Ak. IV, 253–383.

R (RR): *Reflexion(en)* followed by a number relative to their progressive ordering in Kant's *Handschriftlicher Nachlass*, Ak. XIV-XVIII.

Zweig: A. Zweig's Engl. transl. of Kant's *Briefwechsel* (Ak. X-XII), *Kant. Philosophical Correspondence 1759–99*, Chicago, University of Chicago Press, 1967.

[1] See *Kritik der Urtheilskraft*, Ak. V, §59: the intuitions needed to prove the reality of concepts "in the case of empirical concepts [...] are called examples, in that of pure concepts of understanding [...] are called schemata". Cf. also A138 = B177. On this question cf. W.H. Walsh, 'Schematism', *Kant-Studien* **49** (1957), reprinted in R.P. Wolff (ed.), *Kant. A Collection of Critical Essays*. London, Macmillan, 1968, pp. 75–6.

[2] Cf. H.J. Paton, *Kant's Metaphysics of Experience*. 2 vol. London, Macmillan, 1951, I, p. 36.

[3] Cf. *Logik Philippi*, p. 456: definitions are nominal "wenn der Begriff zureichend ausdrückt alles was ich von der Sache erkenne".

[4] The notions of *Umfang* and *Inhalt* are obviously connected with the notions of *étendue* and *compréhension* of Port Royal Logic.

[5] *Logik*, Ak. IX, 61: "Zum Real-Wesen des Dinges (esse rei) die Erkenntniss derjenigen Prädicate erfordert wird, von denen alles, was zu seinem Dasein gehört, als Bestimmungsgründe, abhängt". See also *R* 2321.

[6] Cf. *RR* 2994, 2995.

[7] Cf. *Logik*, Ak. IX, p. 59.

[8] For a very thorough exposition of this point see J. Proust, 'Analyse et définition chez Kant', *Kant-Studien* **66** (1975), especially pp. 17–27.

[9] This privilege of real definitions is clear from the fact that in the case of empirical concepts even an incomplete and imprecise real definition is much more useful (and used) than an equally incomplete and imprecise nominal definition. In the case of categories, although one could try to give them nominal and analytic definitions (A244–5), what is important is the procedure of schematization which shows the way categories refer to objects, and hence shows their reality. One might say that the schematization of categories is a fruitful but only tentative 'real definition' of them (A731 = B759 Footnote).

[10] Already in his earliest writings Kant stressed the 'hypothetical' (i.e. axiomatic) nature of any mathematical system (cf. for example *Gedanken*, Ak. I, 139–40). But he did not let this interfere with his effort to give mathematics an objective foundation, although at first he supported this effort by metaphysics.

[11] L.W. Beck, 'Kant's Theory of Definition', *Philosophical Review* **65** (1956), reprinted in R.P. Wolff (ed.), *Kant. A Collection of Critical Essays, op. cit.*, p. 31.

[12] These fundamental propositions, which are "*augenscheinliche Sätze*", correspond, judging from Kant's examples, to both critical *analytische Grundsätze* and axioms (Ak. II, 281: "Das Ganze ist alle Theilen zusammen genommen gleich; zwischen zewi Punkten kann nur eine gerade Linie sein). Probably this depends on the fact that Kant is referring here to mathematics in general.

[13] *R* 3970: "Alle *principia primitiva* sind entweder elementarsätze und analytisch oder *axiomata* und sind synthetisch. Die rationale sind analytisch, die empirische synthetisch, imgleichen mathematische"; *R* 3738: "Alle analytische Urtheile sind rational und umgekehrt. Alle synthetische Urtheile sind empirisch und umgekehrt . . . "; *R* 3923: "Einige Grundsätze sind analytisch und betreffen das formale der Deutlichkeit in unserer Erkenntnis. Einige sind synthetische und betreffen das materiale, als da find die arithmetischen, geometrischen und chronologischen imgleichen die empirischen".

[14] The turning-point in this transition from extreme empiricism to criticism can be seen in *R* 3932: "Die ideen und regeln der Vernunft werden auch in dem Verhältnisse empirischer Begriffe gebrauchet, und dieses ist ihr natürlicher und richtiger Gebrauch; sie sind alsden aber auf *iudicia empirica primitiva* gegründet, die nur durch induction allegemein sind. Aber eben diese Urtheile der Vernunft, so fern sie rein sind, sollen an sich allgemein seyn". Then Kant asked the fundamental question: (*R* 3926) "Wie werden empirische und synthetische [Urtheile] allgemein?".

[15] See *H. Nach*, pp. 17–18; 29–30. This division of mathematics was common to the major mathematical textbooks of the time, such as C. Wolff, *Auszug aus den Anfangsgründen aller mathematischen Wissenschaften*, Halle 1717 which Kant used for his lectures.

[16] Kant's examples of analytic *Grundsätze* are:

(1) "2 Grössen einer 3. gleich sind gleich sich selbst" (*H. Nach*, p. 23; Ak. XI, 556; *Zweig*, p. 130);

(2) "Gleiches, zu Gleichem hinzugethan oder von diesem abgezogen, in Gleiches gebe" (A164 = B204);

(3) "$a = a$, das Ganze ist sich selber gleich" (B17; *R* 4634; *H. Nach*, p. 23; *Prol.* §2);

(4) "$(a + b) > a$, d.i. das Ganze ist grösser als sein Theil" (B17; *Prol.* §2; Ak. VIII, 196);

(5) "Das Ganze ist allen Theilen zusammen genommen gleich" (Ak. II, 281). Here *Grundsätz* (3) is a special case of Euclid's common notion (4); *Grundsätz* (5) corresponds to an axiom introduced by Clavius. All the other *Grundsätze* correspond exactly to the Euclidean common notions (cf. T. L. Heath's Commentary to his translation of Euclid, *The Thirteen Books of the Elements*, New York, Dover Publications, 1956[2], vol. I, pp. 221–232).

[17] See Ak. XI, 40–8; *Zweig*, p. 145: "The mathematician cannot make the smallest assertion about any object whatsoever without exhibiting it (or, if we are considering only quantities without qualities, as in algebra, exhibiting the quantitative relationships for which the symbols stand) in intuition". Cf. also *R* 13.

[18] "*Gleiches zu Gleichem addiert gibt gleiche Summe*" is a proposition belonging to algebra not to be confused with the analytic *Grundsätz* of the *allgemeine Grössenlehre* listed as (2) in note 16 above. In algebra "addiren" refers to an operation to be performed on numbers, although not on determinate numbers. The "hinzutun" of the analytic *Grundsätz* (2) refers instead to an operation that could be performed, for example, also on spatial (or geometrical) entities and therefore not necessarily on numbers. S. Stenlund has recently maintained that, if the operation of addition is for Kant a non-defined primitive and its properties are exposed as axioms (that define it implicitly), then formulae like $7 + 5 = 12$ are synthetic, where 'synthetic' means 'non-decidable from definitions alone' (see S. Stenlund, 'Analytic and Synthetic Arithmetical Statements', in S. Stenlund (ed.), *Logical Theory and Semantic Analysis*, Dordrecht, Reidel, 1974, pp. 199–211). Now, although Stenlund is right in saying that for Kant addition has an implicit definition, he does not distinguish between algebraic and arithmetical addition. Indeed he seems to have in mind only addition of specific numbers, as in $7 + 5 = 12$, which for Kant has an explicit definition (cf. III.4).

[19] The German text is: (*R* 3973) "Keine andere blos willkürliche Begriffe der reinen Vernunft können in uns entstehen, als die durch die Wiederholung, folglich der Zahlen und Grösse".

[20] Original text: (*R* 3940) "In allem Erkenntnisse der Vernunft sind nur Verhältnisse zu betrachten, und diese sind entweder gegebene [. . .] oder gedichtet. Dichten aber können wir keine Verhältnisse, (von) deren Möghlichkeit wir überzeugt seyn können, als den Grösse nach durch Wiederholung in der Zahlwissenschaft"; (*R* 3969) "Unsere Vernunft enthält nichts als *relationes*. Wenn nun diese nicht gegeben sind durch die Verhältnisse nach Raum und Zeit in der Erfahrung, auch nicht durch die Wiederholung

und die Zusammensetzung des Eins aus vielen bey der reinen Mathematik, so sind sie keine *relationes*, welche auf objecte gehen, sondern nur Verhältnisse unserer Begriff nach Gesetzen unserer Vernunft".

[21] Se *H. Nach*, p. 21: "Ein Ganzes, aus vielen Dingen von einerlei Art, deutlich ausgedrückt, ist Zahl".

[22] For the pre-critical connection between number, repetition and measure see *H. Nach*, p. 17: "das Maas der Grösse (Vielheit) ist die Einheit, oder Eins"; "wenn ich deutlich ausdrücke, wieviel in einer Sache das Eins enthalten, so ist das eine Zahl".

23 Here the term 'addiren' has a definite meaning and, in this sense, constitutes the arithmetical interpretation of the term 'hinzutun' used in algebra.

[24] My argument is related to that of D. Prawitz in *Meaning and Proofs: on the Discussion of the Conflict between Classical and Intuitionistic Viewpoints*, Institute of Philosophy, University of Oslo, 1976, pp. 24 ff. Prawitz gives very significant examples involving, for instance, the commutativity law of addition, while in Kant we find only less relevant uses of algebraic laws, such as that quoted in the text. Nevertheless it is interesting to note that a former student of Kant, J. Schultz, while remaining faithful to Kantian philosophy, provided an axiomatization of algebra in *Prüfung der Kantischen Kritik der reinen Vernunft*, Königsberg 1789–1792, and in other writings on mathematics. His axioms include the commutativity law of addition. G. Martin, who first become interested in Schultz's works, in *Arithmetik und Kombinatorik bei Kant* (Itzehoe 1938), Berlin, De Gruyter, 1972, goes so far as to maintain that Schultz was directly influenced by Kant on these questions. But, whether Martin is right or wrong, his thesis is unnecessary for the soundness of my argument.

[25] See G. Knauss, 'Extensional and Intensional Interpretation of synthetic propositions *a priori*', in L.W. Beck (ed.), *Proceedings of the Third International Kant Congress*, Dordrecht, Reidel, 1972, p. 358: "the number-formulae form an infinite set of individuals which cannot be deduced from each other".

[26] Cf. A31 = B47: time "has only one dimension; different times are not simultaneous but successive [. . .]". In the *Inaugural Dissertation* (Ak. II, 397) space is the subject-matter of geometry and time is the subject-matter of pure mechanics, not of arithmetic. Cf. also *Prol.* §10.

[27] In the *Prize Essay* (Ak. II, 277) Kant mentions, in addition to space, another mathematically undefinable concept, namely similarity (*Ähnlichkeit*). Indeed he insists that one should not follow C. Wolff's attempt to provide a mathematical definition for it. Now, while E.W. Beth (*The Foundations of Mathematics*, North-Holland, Amsterdam, 1959, p. 45) believes that Kant misses a good chance here to do something 'salutary' for mathematics, G. Martin in *I. Kant. Ontologie und Wissenschaftstheorie* (Berlin, De Gruyter, 1969, p. 309) seems convinced that Kant considers similarity to be implicitly defined by its properties. However, since Kant does not expound such properties, it seems more reasonable to say that he had in mind simply a 'common notion' of similarity and did not consider it to be either explicitly or implicitly definable.

[28] On this question see K. D. Wilson, 'Kant on Intuition', *The Philosophical Quarterly* **25**, 252ff. (1975).

[29] Cf. A25 = B39: the parts of space "cannot precede the one all-embracing space as being, as it were, constituents out of which it can be composed; on the contrary they can be thought only as *in* it". See also *Inaugural Dissertation*, Ak. II, 403 *Note*, and

A169 = B211. The pre-critical *R* 3926 considers the concepts of 'part' and 'whole' – along with the concepts of unity, plurality, space and time - as two of the basic notions of ontology. Therefore the part-whole relation may be included among the "few fundamental concepts of space" that, according to the *Prize Essay* (Ak. II, 282), mediate the application of universal *Grössenlehre* to geometry. But also in pre-critical writings space was prior to traditional primitive notions.

[30] "Kant emphasized that space was an individual, the notion of which, understood in a way analogous to ostension, and the same ostensive understanding, would be necessary for the [...] primitives of geometry" (C. Parsons, 'Kant's Philosophy of Arithmetic', in S. Morgenbesser, P. Suppes, M. White (eds.), *Philosophy, Science and Methodology*, New York, St. Martin's Press, 1969, p. 574). Kant linked geometrical primitives to knowledge of an immediate kind even before developing the Transcendental Aesthetic, i.e. when he was still speaking of a *recta ratio* which enables us to single out such primitives (see e.g. Ak. I, 394, 403, 409; Ak. II, 11, 16). On this last point cf. G. Tonelli, *Elementi metodologici e metafisici in Kant dal 1745 al 1768*, Turin, Edizioni di Filosofia, 1959, p. 148.

[31] See T.L. Heath's Commentary to Euclid, *op. cit.*, vol. I, p. 232. Kant does not give a detailed list of geometrical axioms, but only mentions some of them. For instance, in addition to the axiom quoted in the text: (A) "Between two points there can be only one straight line" (Ak. II, 281; *Inaugural Dissertation*, §15 C; A163 = B204, A300 = B356; this axiom corresponds to Euclid's first postulate and presupposes the definition of 'straight line'); (B) "In a triangle the sum of two sides is greater than the third" (A25 = B39; in Euclid this is Proposition 20 of Book I); (C) "Space has three dimensions" (Ak. II, 281; *Inaugural Dissertation* §15 C); D) "Three points lie always in a plane" (A732 = B761).

[32] Euclid's third Postulate states: "To describe a circle with any center and distance". See Note 34 below.

[33] This depends also on the strict connection between arithmetic and logic. The latter was identified by Kant with the science of expressing the true laws of human thinking and was therefore *unique*.

[34] Letter to K.L. Reinhold (Ak. XI, 40–48; *Zweig*, p. 146). In a letter to Marcus Herz (Ak. XI, 48–55; *Zweig*, p. 155) Kant says: "I may always draw a circle free hand on the board and put a point in it, and I can demonstrate all properties of the circle just as well in it, presupposing the (so-called) nominal definition, which in fact is a real definition, even if this circle is not at all like one drawn by rotating a straight line attached to a point". Furthermore Kant considers the proposition "to inscribe a circle" as a "practical corollary of the definition (or so-called postulate)", and indeed this proposition corresponds to Euclid's third postulate (cf. Note 32). In Kant's own terminology, then, postulates are accessories, as it were, of definitions losing part of their ancient prestige. Further evidence for the irrelevance of the exterior form of mathematical definitions is provided by the *Prize Essay* where, on the same page (Ak. II, 278), a strictly genetic definition and a non-genetic one are given by Kant as paradigmatic instances of mathematical definitions.

[35] In arithmetic, not only definitions, but also postulates are universal only with respect to their use. For instance propositions like 7 + 5 = 12, which are called postulates by Kant (Ak. XI, 555–6; *Zweig*, pp. 129–30), have this kind of universality (A164–B205).